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Discrete model for the stability of continuous welded rail

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Summary

In this paper a discrete model is developed for the buckling analysis of continuous welded rail subjected to temperature load. The model is based on a nonlinear analysis in total lagrangean formulation. The structure consists of beam elements and lateral, longitudinal and torsional spring elements. The source of nonlinearity is due to the geometric nonlinearity of the rail high axial forces and also to the nonlinearity of material type for the lateral and longitudinal resistance of the ballast and the torsional resistance of the fasteners. The use of a displacement control algorithm leads the analysis beyond the critical point and permits a more realistic computation of the structural safety. The track model is encoded into a special purpose program which allows a parametric study of the influence of vehicle loading, the stiffness properties of the structure and of the geometric imperfections on the track stability.

The validity of the present model is verified through a series of comparative analyses with other author's results.

KEYWORDS: Continuous welded rail, Non-linear stability analysis, Temperature loading, Structural safety.

1. INTRODUCTION

The first computational models of the buckling of the continuous welded rail (CWR) were developed at the beginning of the 1930 years. These models can take into account the main parameters which control the stability of CWR like the horizontal and vertical stiffness of the rail, the longitudinal and transversal resistance of the rail, the torsional resistance of the fasteners, the stresses induced by the vehicle and temperature loading, the geometry and the misalignment of the rail. In the SCFJ model presented in this paper the structure consists of beam elements and lateral, longitudinal and torsional spring elements. The beam elements are modeling the rail and have geometric nonlinear characteristics due to high compressive thermal stresses. The spring elements are describing the material nonlinear behavior of the ballast and the fasteners.

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2. DEVELOPMENT OF THE TRACK MODEL

2.1. The longitudinal ballast behavior

In the SCFJ model the longitudinal resistance of the ballast is introduced by spring elements having the, linear or bilinear displacement-force curves given in figure 1. In the case of vehicle loading, the bilinear curve is corrected [6] by the equation (1) taking into account the vertical force Q on each sleeper.

$$U_v^c = U_v + Q \cdot tan\,\phi_L \quad , \quad U_v^c \geq \frac{2}{3} U_v \tag{1}$$

In the above equation U_v is the reference value of the longitudinal resistance (without vehicle loading), U_v^c is the corrected value of this resistance and ϕ_L is the angle of the longitudinal friction between the sleeper and the ballast.

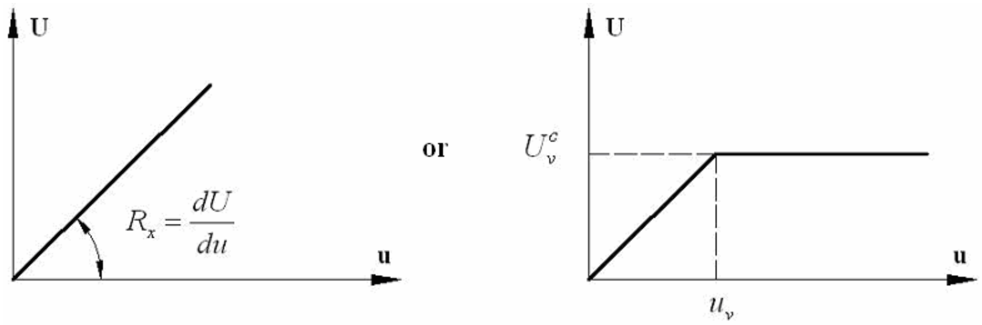

Figure 1. Longitudinal resistance versus longitudinal displacement of the ballast

2.2. The transversal ballast behavior

The transversal resistance of the ballast is introduced by spring elements having the displacement-force curves given in figure 2. In both cases the elasto-plastic model includes softening. This kind of ballast behavior has been measured for consolidated ballast. In the case of vehicle loading, the bilinear curve is corrected [6] by the equation (1) taking into account the vertical force Q on each sleeper using equations (2), (3) or (4).

$$V_v^c = V_v + Q \cdot tan\,\phi_T \quad , \quad V_v^c \geq \frac{2}{3} V_v \tag{2}$$

$$V_r^c = V_r \cdot V_v^c / V_v \tag{3}$$

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$$\text{for } tipV{=}1:\ V = V_r^c + (V_v^c - V_r^c)\cdot 2^{-2\frac{v-v_v}{v_r-v_v}} \quad (\text{if } v > v_v) \tag{4}$$

In the above equations V_v is the reference value of the peak transversal resistance (without vehicle loading), V_v^c is the corrected value of this resistance, ϕ_T is the angle of the transversal friction between the sleeper and the ballast, V_r is the reference value of the residual transversal resistance (without vehicle loading), and V_r^c is the corrected value of this resistance. In the case of exponential softening the difference $V_v^c - V_r^c$ is half at the middle of $v_v^c - v_r^c$ interval.

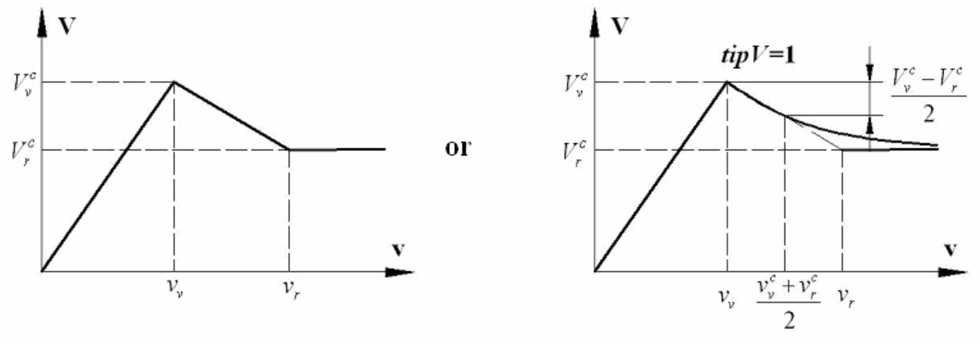

Figure 2. Transversal resistance versus transversal displacement of the ballast

2.3. The torsional stiffness of the fasteners

The resistance of the fasteners is introduced by torsional springs having the linear or tri-linear behavior shown in figure 3. In the case of loaded rail this behavior also can be corrected taking into account the vertical force acting on each sleeper.

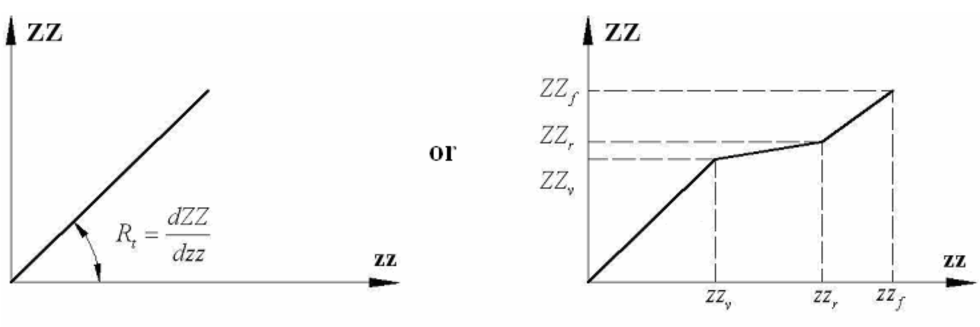

Figure 3. The torsional stiffness of the fasteners

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2.4. The geometrical and physical characteristics of the rail

The rail is modeled by beam elements having area of the cross section A, second order moment about the vertical and horizontal axes I_z and I_y respectively. The Young modulus and the thermal expansion coefficient of the material are E and respectively α. In the model the misalignment of the rail can be described by two types of curves: a complete or a half cosine wave having the total length λ and the amplitude δ (figure 6). The length of the model is an input of the program. At the end of the model special infinite boundary elements are introduced -equivalent with the theoretical infinite rail [6]. This elements lead to the reduction of the length of the model and hence the computational effort. Further reduction can be obtained by using the symmetric half structure.

3. THE NUMERICAL ALGORITHM

Since in a simplified manner, the horizontal and vertical behavior are considered decoupled, the numerical algorithm has two phases.

3.1. The computational model for vertical loadings

This model is linear elastic consisting of a beam on elastic springs. The nodes of the structure are considered at the sleepers. Each node has two degrees of freedom: the vertical translation w and the rotation θ_y. The system of equations of equilibrium is:

$$\mathbf{Ka} = \mathbf{F}. \tag{5}$$

where:

K is the stiffness matrix of the structure and results by assembling the stiffness matrices **k** of the beams and the vertical stiffness of the fasteners.

a is the displacement vector of the nodes of the structure.

F is the vector of forces at the nodes of the structure, which (in this case) results by assembling the vectors $\mathbf{f}_0$ of the forces on the beams.

The stiffness matrix $\mathbf{k}_{(4x4)}$ of a beam is given by the equation:

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Here $\mathbf{B}_{(2\times 4)}$ is a transformation vector, which links the vector of displacements of the beam and the reduced vector of displacements of the beam. The reduced vector of displacements does not contain the rigid body displacements.

$$\mathbf{a}_{el}^{d} = \begin{Bmatrix} \theta_{yi}^{d} \\ \theta_{yi+1}^{d} \end{Bmatrix} = \mathbf{B} \cdot \mathbf{a}_{el} = \begin{bmatrix} 1/L & 1 & -1/L & 0 \\ 1/L & 0 & -1/L & 1 \end{bmatrix} \begin{pmatrix} w_i & \theta_{yi} & w_{i+1} & \theta_{yi+1} \end{pmatrix}^T \tag{7}$$

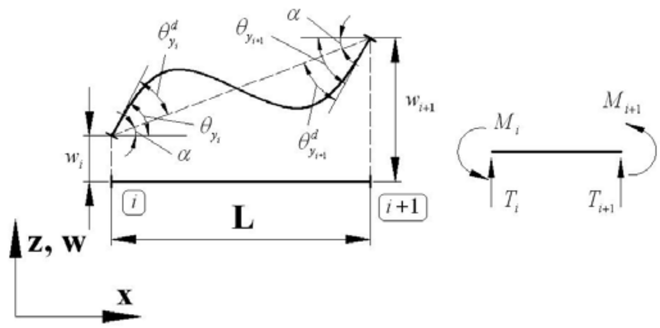

Figure 4. The displacements of the beam in the vertical plane

$\mathbf{k}^{d}_{(2\,x\,2)}$ is the reduced stiffness matrix of the beam.

$$\mathbf{k}^{d} = \frac{EI_y}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \tag{8}$$

If the beam is loaded, the vector $\mathbf{f}_0$ of equivalent forces in the nodes is given by equations (9).

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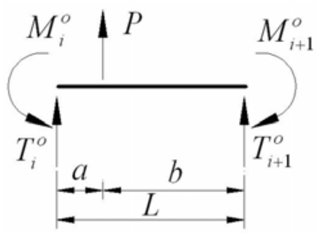

Figure 5. Equivalent nodal forces of the beam

The stiffness matrices and the load vectors of the beams are assembled by the relation (10).

$$\mathbf{K}_{ind,ind} = \mathbf{K}_{ind,ind} + \mathbf{k}, \quad \mathbf{F}_{ind} = \mathbf{F}_{ind} + \mathbf{f}_0 \,. \tag{10}$$

Here *ind* is the vector of the indices of the displacements of the current beam.

The stiffness of the sleepers is assembled with the help of the equation (11).

$$\mathbf{K}_{jnd,jnd} = \mathbf{K}_{jnd,jnd} + R_z L \tag{11}$$

In the above equation *jnd* is the set of indices of vertical displacements of the nodes *jnd*=1, 3, ..., 2*nnd*-1. The constraints of the structure are introduced by setting to zero the displacements of the supports. The free displacements of the nodes result by solving the system of linear equations:

$$\mathbf{a}_{id} = (\mathbf{K}_{id,id})^{-1} \mathbf{F}_{id} . \tag{12}$$

In equation (12) *id* is the set of the free displacements of the structure.

Using the vertical displacements, the vertical force on each sleeper can be computed by the equation (13)

$$Q = -wR_z L + G_{sleeper} . \tag{13}$$

The transversal, longitudinal and torsional resistances are corrected taking into account the forces Q on each sleeper using equations (1) to (4).

3.2. The computational model in the horizontal plane

The model is a straight or curved beam on elastic supports with misalignments (figure 6). The nodes of the structure are considered at the sleepers. At each node are introduced longitudinal, transversal and rotational spring elements which are modeling the sleepers. The infinite boundary elements at the ends of the model have equivalent characteristics (Young modulus and thermal expansion coefficient) in order to replace the theoretical infinite rail [6]. The loading of the model is an increase of the temperature in the rail. The characteristics of the beams and of the springs correspond to the two rails of the track panel. A node has three degrees of freedom: two linear displacements in the horizontal plane, u and v and the rotation θ_z around the vertical axis. In the analysis of the structure the goal is to obtain the displacement-temperature curve. The problem is solved by a displacement control based incremental process. The behavior of the system is determined as a sequence of increments of state parameters (forces and displacements). In the current increment j characterized by the small control displacement δv_{cj}, the nonlinear behavior of the system can be approximated by a linear relation between the successive increments of the state parameters:

$$\mathbf{a}_{j+1} = \mathbf{a}_j + \delta\mathbf{a}_j , \quad \delta\mathbf{F}_j = \mathbf{K}_j \delta\mathbf{a}_j . \tag{14}$$

In the above equation $\mathbf{a}_j$ is the displacement vector in the current configuration, $\delta\mathbf{a}_j$ is the increment of the displacements, $\delta\mathbf{F}_j$ is the incremental load vector and $\mathbf{K}_j$ is the incremental (tangent) stiffness matrix of the structure.

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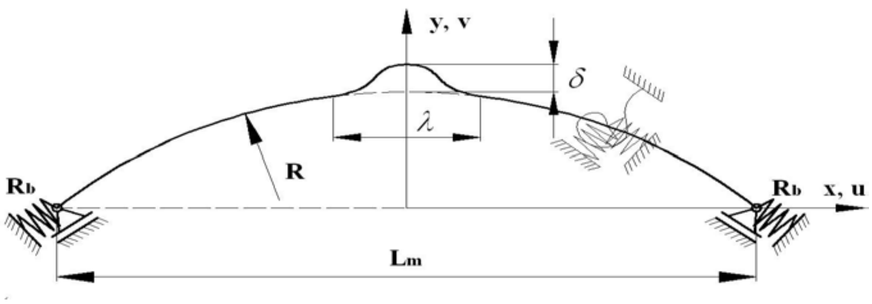

Figure 6. The model for horizontal displacements

By using equations (14), the following incremental scheme results:

$$\delta \mathbf{a}_j = (\mathbf{K}_j)^{-1}\delta \mathbf{F}_j, \quad \mathbf{a}_{j+1} = \mathbf{a}_j + \delta \mathbf{a}_j . \tag{15}$$

In this paper an improved scheme is used, known as Heun's or midpoint rule:

$$\begin{aligned} &\mathbf{a}_{j+1/2} = \mathbf{a}_j + 1/2(\mathbf{K}_j)^{-1}\delta \mathbf{F}_j, \\ &\mathbf{K}_{j+1/2} = \mathbf{K}(\mathbf{a}_{j+1/2}), \\ &\delta \mathbf{F}_{j+1/2} = \delta \mathbf{F}(\mathbf{a}_{j+1/2}), \\ &\mathbf{a}_{j+1} = \mathbf{a}_j + 1/2(\mathbf{K}_{j+1/2})^{-1}\delta \mathbf{F}_{j+1/2}. \end{aligned} \tag{16}$$

The incremental load vector $\delta \mathbf{F}_j$ is not computed. The incremental displacement $\delta \mathbf{a}_j$ is the result of a yet unknown increment of the temperature produced by a known increment δv_{cj} of the control displacement. For simplicity, in the next equations indices j of the current configuration are dropped. The displacement control consists of loading the system with displacement increments δv_c in a specific node. As a rule in this paper: the control displacement is taken as the maximum transversal displacement of the node on the symmetry axe of the structure. The phases of the computation are the following:

- It is adopted a base system with the control displacement fixed at zero.
- This base system is loaded with two load cases:

i) Load 1 is a temperature increase $\delta T = 1$, which produces displacements $\delta a^{(1)}$ at the free nodes and reaction $R^{(1)}$ in the artificial support.

ii) Load 2 is a displacement δv_c of the artificial support, which produces displacements $\delta \mathbf{a}^{(2)}$ at the free nodes and reaction $R^{(2)}$ in the artificial support.

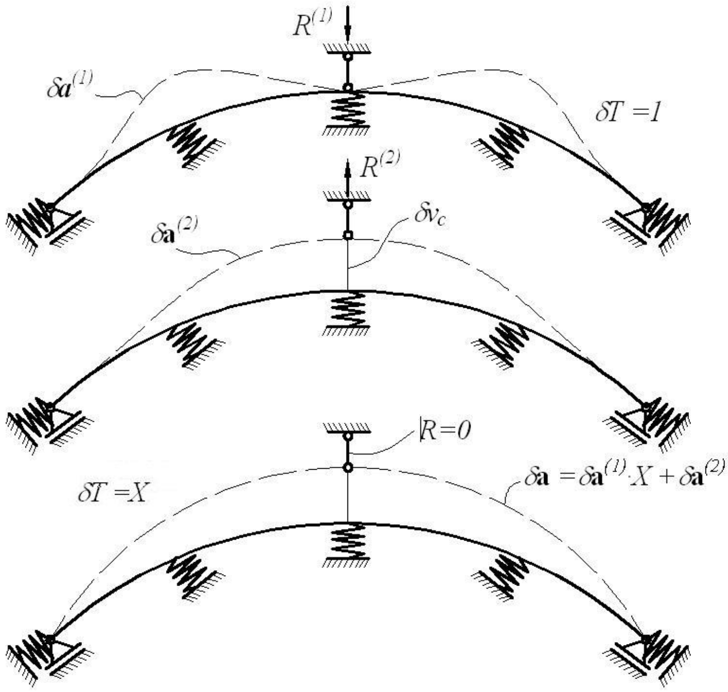

Figure 7. The determination of the temperature and displacement increments

The base system and the initial system are identical if the total reaction in the artificial support is zero. $R = R^{(1)}\delta T + R^{(2)} = 0$. This equation yields the unknown variation δT of temperature and the incremental displacements $\delta \mathbf{a}$ of the free nodes.

$$\delta T = -R^{(2)} / R^{(1)}, \tag{17}$$

$$\delta \mathbf{a} = \delta \mathbf{a}^{(1)} \cdot \delta T + \delta \mathbf{a}^{(2)}. \tag{18}$$

The tangent stiffness matrix $\mathbf{K}_j$ in the j increment depends on the parameters of the system in the current step and results by assembling the stiffness matrices $\mathbf{k}_{t(6\,x\,6)}$ of the beams and of the springs which model the sleepers.

$$\mathbf{k}_t = EA/L \cdot \mathbf{r}^T \mathbf{r} + \mathbf{B}^T (\mathbf{k}^d + \mathbf{k}_G^d) \mathbf{B} + N_j / L \cdot \mathbf{z}^T \mathbf{z}. \tag{19}$$

In this equation $\mathbf{r} = (-\cos\beta \ \ -\sin\beta \ \ 0 \ \ \cos\beta \ \ \sin\beta \ \ 0),$
$z = (\sin\beta \ \ -\cos\beta \ \ 0 \ \ -\sin\beta \ \ \cos\beta \ \ 0),$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{L}\begin{bmatrix} \mathbf{z} \\ \mathbf{z} \end{bmatrix}, \quad \mathbf{k}^d = \frac{EI_z}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \quad \mathbf{k}_G^d = \frac{N_j L}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix},$$

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and N_j is the axial force in the beam in the j-th incremental step: $N_j=EA\ \Delta L_j/L_j$, $\Delta L_j = L_j - L_0$, $L_0 = \sqrt{(\mathbf{x}_{i+1}^0 - \mathbf{x}_i^0)^2}$, $L_j = \sqrt{(\mathbf{x}_{i+1}^j - \mathbf{x}_i^j)^2}$.

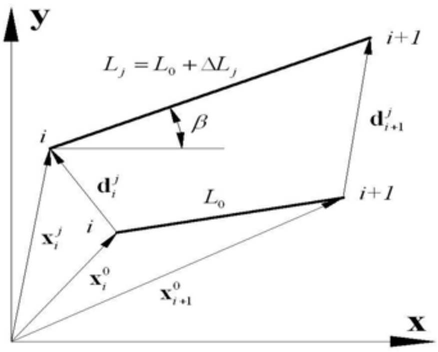

Figure 8. The axial deformation of the beam

Matrices $\mathbf{k}^d$ and $\mathbf{k}_G^d$ are the material and geometric stiffness matrices respectively. They are expressed with the reduced set of displacements which produce deformations and they are not containing the rigid body displacements of the beam. This reduced form of the stiffness matrices needs less computational effort and speeds up significantly the computation. Equation (19) introduces the non-linear effect of the axial force N_j. The complete tangent stiffness matrix in the updated lagrangean formulation used here has two more terms corresponding to the variation of the length of the beam in bending and to the effect of the shear force [1], [2], [3], [4], [5]. Since in the current cases the structure is divided in a sufficient number of beams, the errors are very small, when neglecting these two terms. In a study using the complete tangent stiffness matrix and equation (19) the differences between the resulting limit temperatures were only at the fifth digit.

4. NUMERICAL EXPERIMENTS

The validity of the present model is verified through a series of comparative analyses with other author's results. The numerical example presented here corresponds with one given in [1]. The track length is L=24.359/2 m corresponding to 21 sleepers on the symmetric half of the structure. The curvature radius is R=400 m. The horizontal misalignment is characterized by a half wave cosine with a length λ=9.144 m and an amplitude δ=0.0381 m. The rails have the characteristics of two AREA 136 rails. The vertical stiffness of the ballast elements is R_z=68900 kN/m per meter of track. The longitudinal stiffness is R_x=1378 kN/m per meter of track. The torsional stiffness of the fasteners is 111.250 kNm/rad per meter of track. Laterally the ballast is modeled by the tri-linear constitutive behavior given in figure 2. The reference values of the lateral peak resistance and residual resistance are V_v=17.508 and V_r=9.630 kN per meter of track. These values are corrected with the vertical forces resulting from the vehicle loading. The lateral displacement at the peak value is v_v=0.00635 m and at the limit value is v_r=0.0381 m. The model is vertically loaded by a vehicle with two bogies represented by four vertical loads of 293 kN each. The centre spacing between the bogies is 12.85 m. The spacing between the axles in a bogie is 1.78 m. The centre of the misalignment is located in the middle between the bogies. The track is loaded by a temperature increase from zero to a maximal value corresponding to the buckling of the rail. The lateral displacement of the middle node versus the resulting temperature increase is shown in figure 9.

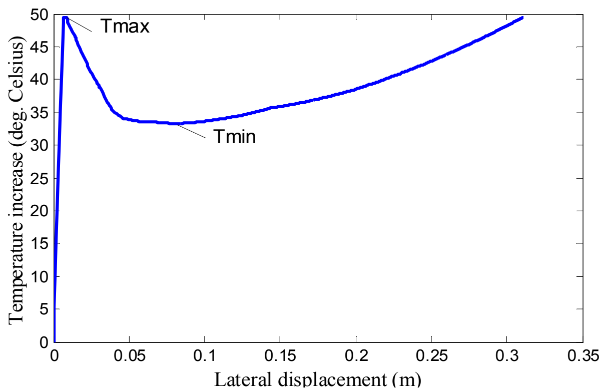

Figure 9. Lateral displacement versus temperature increase

The curve in the figure 9 is characterized by two points: T_{max} - the maximum increase of temperature for which the buckling certainly starts, and T_{min} - the minimum increase of temperature which occurs in the post-buckling domain. The values computed by the SCFJ model - T_{max}=49.5 ^{0}C and T_{min}=33.3 ^{0}C - are in a good agreement with those given in [1].

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